

## Chapter 4

### PRINCIPLE OF MATHEMATICAL INDUCTION

#### INTRODUCTION

To prove certain results or statements in Algebra, that are formulated in terms of  $n$ , where  $n$  is a natural number, we use a specific technique called principle of mathematical induction (P.M.I)

#### **Steps of P.M.I**

**Step I** - Let  $p(n)$ : result or statement formulated in terms of  $n$  (given question)

**Step II** – Prove that  $P(1)$  is true

**Step III** – Assume that  $P(k)$  is true

**Step IV** – Using step III prove that  $P(k+1)$  is true

**Step V** - Thus  $P(1)$  is true and  $P(k+1)$  is true whenever  $P(k)$  is true.

Hence by P.M.I,  $P(n)$  is true for all natural numbers  $n$

#### **Type I**

Eg: Ex 4.1

1) Prove that

$$1+3+3^2+\dots\dots\dots+3^{n-1} = \frac{3^n-1}{2}$$

Solution:-

**Step I** : Let  $P(n)$ :  $1+3+3^2+\dots\dots\dots+3^{n-1} = \frac{3^n-1}{2}$

**Step II**:  $P(1)$ :

$$\text{LHS} = 1$$

$$\text{RHS} = = \frac{3-1}{2} = \frac{2}{2} = 1$$

$$\text{LHS}=\text{RHS}$$

Therefore  $p(1)$  is true.

**Step III**: Assume that  $P(k)$  is true

$$\text{i.e } 1+3+3^2+\dots\dots\dots+3^{k-1} = \frac{3^k-1}{2} \quad \text{_____ (1)}$$

**Step IV:** we have to prove that  $P(k+1)$  is true.

$$\text{ie to prove that } 1+3+3^2+\dots+3^{k-1} + 3 = \frac{3^{k+1}-1}{2}$$

**Proof**

$$\text{LHS} = (1+3+3^2+\dots+3^{k-1}) + 3$$

$$= \frac{3^k-1}{2} + 3^k \text{ from eq(1)}$$

$$= \frac{3^k-1 + 2 \cdot 3^k}{2}$$

$$= \frac{3 \cdot 3^k - 1}{2} = \frac{3^{k+1}-1}{2} = \text{RHS}$$

Therefore  $P(k+1)$  is true

**Step V:** Thus  $P(1)$  is true and  $P(k+1)$  is true whenever  $P(k)$  is true. Hence by

P.M.I,  $P(n)$  is true for all natural number  $n$ .

### **Text book**

#### **Ex 4.1**

Q. 1,2, 3\*\*(HOT), 4, 5\*,6\*,7,8,9,10\*,11\*\*,12,13\*\*,14\*\*,15,16\*\*,17\*\*,  
eg 1, eg 3

#### **Type 2**

Divisible / Multiple Questions like Q. 20\*\*,21,22\*\*,23 of Ex 4.1

eg 4, eg 6\*\*(HOT)

Q 22. Prove that  $3^{2n+2}-8n-9$  is divisible by 8 for all natural number  $n$ .

**Solution**

**Step I:** Let  $p(n): 3^{2n+2}-8n-9$  is divisible by 8

**Step II:**  $P(1): 3^4 - 8 - 9 = 81 - 17 = 64$  which is divisible by 8

Therefore  $p(1)$  is true

**Step III:** Assume that  $p(k)$  is true

$$\text{i.e } 3^{2k+2} - 8k - 9 = 8m; \quad m \text{ is a natural number.}$$

$$\text{i.e } 3^{2k} \cdot 9 = 8m + 8k + 9$$

$$\text{ie } 3^{2k} = \frac{8m + 8k + 9}{9} \quad \text{_____ (1)}$$

**Step IV:** To prove that  $p(k+1)$  is true.

ie to prove that  $3^{2k+4} - 8(k+1) - 9$  is divisible by 8.

$$\text{Proof: } 3^{2k+4} - 8k - 17 = 3^{2k} \cdot 3^4 - 8k - 17 = \left(\frac{8m+8k+9}{9}\right) \times 3^4 - 8k - 17 \text{ (from eqn (1))}$$

$$= (8m+8k+9)9 - 8k - 17 = 72m + 72k + 81 - 8k - 17 = 72m - 64k + 64 = 8[9m - 8k + 8] \text{ is divisible by 8.}$$

**Step V:** Thus  $P(1)$  is true and  $P(k+1)$  is true whenever  $P(k)$  is true. hence by P.M.I,  $P(n)$  is true for all natural numbers  $n$ .

**Type III:** Problems based on Inequations

Ex 4.1 Q. 18,14, eg 7

(Q 18) Prove that  $1+2+3+\dots+n < \frac{(2n+1)^2}{8}$

**Step I :** Let  $P(n): 1+2+3+\dots+n < \frac{(2n+1)^2}{8}$

**Step II:**  $P(1): 1 < \frac{9}{8}$  which is true, therefore  $p(1)$  is true.

**Step III:** Assume that  $P(k)$  is true.

$$\text{ie } 1+2+3+\dots+k < \frac{(2k+1)^2}{8} \text{ (1)}$$

**Step IV:** We have to prove that  $P(k+1)$  is true. ie to

$$\text{prove that } 1+2+3+\dots+k+(k+1) < \frac{(2k+3)^2}{8}$$

**Proof:** Adding  $(k+1)$  on both sides of inequation (1)

$$1+2+3+\dots+k+(k+1) < \frac{(2k+1)^2}{8} + (k+1)$$

$$= \frac{(4k^2+4k+1)+8k+8}{8}$$

$$= \frac{4k^2+12k+9}{8}$$

$$= \frac{(2k+3)^2}{8}$$

$$\text{Therefore } 1+2+3+\dots+k+(k+1) < \frac{(2k+3)^2}{8}$$

$P(k+1)$  is true.

**Step V:** Thus  $P(1)$  is true and  $P(k+1)$  is true whenever  $P(k)$  is true. Hence by P.M.I,  $P(n)$  is true for all natural number  $n$ .

### HOT/EXTRA QUESTIONS

Prove by mathematical induction that for all natural numbers  $n$ .

- 1)  $a^{2n-1} - 1$  is divisible by  $a-1$  (type II)
- 2)  $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$  is an integer (HOT)
- 3)  $\sin x + \sin 3x + \dots + \sin (2n-1)x = \frac{\sin^2 nx}{\sin x}$  (HOT Type 1)
- 4)  $3^{2n-1} + 3^n + 4$  is divisible by 2 (type II)
- 5) Let  $P(n)$ :  $n^2 + n - 19$  is prime, state whether  $P(4)$  is true or false
- 6)  $2^{2n+3} \leq (n+3)!$  (type III)
- 7) What is the minimum value of natural number  $n$  for which  $2^n < n!$  holds true?
- 8)  $7^{2n} + 2^{3n-3} \cdot 3^{n-1}$  is divisible by 25 (type II)

### Answers

- 5) false
- 7) 4